Second-harmonic generation in optical fibers on a continuous-wave background

Weina Cui,[1,2] Guoxiang Huang,[2,3,*] and Bambi Hu[3,4]
[1]*Department of Applied Physics, Nanjing University of Science and Technology, Nanjing 210094, China*
[2]*Department of Physics, East China Normal University, Shanghai 200062, China*
[3]*Department of Physics and Center for Nonlinear Studies, Hong Kong Baptist University, Hong Kong, China*
[4]*Department of Physics, University of Houston, Houston, Texas 77204, USA*
(Received 3 September 2003; revised manuscript received 28 July 2004; published 5 November 2004)

We predict that a different type of second-harmonic generation (SHG) in nonlinear optical fibers is possible for the waves excited from a continuous-wave background. We show that in a normal dispersion regime and near the zero-dispersion point of a single-mode optical fiber the phase-matching condition of the SHG can be satisfied by suitably choosing the wave vectors and frequencies of fundamental and second-harmonic waves. Using a multiscale method the nonlinearly coupled envelope equations for the SHG are derived and their explicit solutions are provided and checked by numerical simulation.

DOI: 10.1103/PhysRevE.70.057602 PACS number(s): 42.65.Tg, 42.65.Ky, 42.81.Dp

Since its first observation the second-harmonic generation (SHG) in optical fibers [1] has attracted much attention (see Ref. [2] and references therein). SHG (as well as other second-order parametric processes) was unexpected theoretically in an electrodipole approximation because the second-order susceptibility $\chi^{(2)}$ vanishes for centrosymmetric materials such as silica. The phenomenon observed may be due to the quadrupole interaction or processes near the material surface where the centrosymmetry is broken. Experimentally the SHG in a Ge-doped silica optical fiber with conversion efficiency up to 10% after irradiation by a laser beam for several hours has been realized [3]. There are numerous works devoted to the experimental and theoretical study on this phenomenon, but up to now the physical mechanism of the SHG in optical fibers remains unclear [4].

In this work we propose a different type of SHG in optical fibers. Such SHG can be realized based on excited waves on a continuous-wave (cw) background without need of any breaking of the centrosymmetry. The idea is as follows. If we consider a nonlinear optical fiber ($\chi^{(2)}=0$) working in the normal dispersion regime, a plane wave (i.e., the cw background) is modulationally stable. An excitation created from the background obeys a set of equations with a second-order nonlinearity. Assuming that the system works near the zero-dispersion (ZD) point, the third-order dispersion makes the linear dispersion relation of the excitation display two branches, which provides the possibility for satisfying the phase-matching condition of the SHG by suitably choosing the wave vectors and frequencies of fundamental and second-harmonic waves. We note that although the nonlinear dynamics of dark solitons generated from a cw background in optical fibers near the ZD point has been investigated intensively [5,6], a possible resonant interaction between these excitations has been overlooked.

Using slowly varying envelope and paraxial approximations, the dimensionless envelope amplitude $u(z,t)$ of the electric field pulse in the neighborhood of the ZD point in a single-mode optical fiber satisfies the modified nonlinear Schrödinger equation [5,6]

$$iu_z - \tfrac{1}{2}\alpha u_{tt} + |u|^2 u = i\beta u_{ttt} \tag{1}$$

where the subscripts z and t mean the partial derivatives. Time t in the reference frame moving with group velocity is measured with the unit of pulse duration T and the longitudinal coordinate z is measured by the unit $T/k^{(1)}$. The parameters $\alpha=k^{(2)}/(Tk^{(1)})$ and $\beta=k^{(3)}/(6T^2k^{(1)})$ denote the dimensionless second- and third-order dispersions of the fiber, respectively. Here $k=k(\omega)$ is the propagation constant and $k^{(j)}=\partial^j k/\partial\omega^j$ ($j=1,2,3$).

Equation (1) has the cw solution $u=u_0\exp(i|u_0|^2 z)$ with u_0 being an arbitrary constant. Without loss of generality we take u_0 to be real. Assuming that $u=u_0[1+a(z,t)]\exp[iu_0^2 z + i\phi(z,t)]$, where $a(z,t)$ and $\phi(z,t)$ denote an excitation from the cw background, Eq. (1) becomes

$$-\phi_z + 2u_0^2 a - \frac{\alpha}{2}a_{tt} + \beta\phi_{ttt} - a\phi_z + \frac{\alpha}{2}\phi_t^2 + 3u_0^2a^2 + 3\beta a_{tt}\phi_t + 3\beta a_t\phi_{tt} + \beta a\phi_{ttt} + \frac{\alpha}{2}a\phi_t^2 + u_0^2a^3 - \beta\phi_t^3 - \beta a\phi_t^3 = 0, \tag{2}$$

$$a_z - \frac{\alpha}{2}\phi_{tt} - \beta a_{ttt} - \alpha a_t\phi_t - \frac{\alpha}{2}a\phi_{tt} + 3\beta\phi_t\phi_{tt} + 3\beta a_t\phi_t^2 + 3\beta a\phi_t\phi_{tt} = 0. \tag{3}$$

This set of nonlinear coupled equations are of *quadratic* nonlinearity, a property already used in the derivation of a Korteweg–de Vries equation for studying the dynamics of dark solitons in optical fibers [5,6].

Based on Eqs. (2) and (3), an analysis of the linear stability of the cw solution against a small perturbation shows that the cw solution is modulationally stable if the fiber is working in the normal dispersion regime (i.e., $\alpha>0$). Assuming a and ϕ varying with the form $\sim\exp(i\omega t - ikz)$ we get the linear dispersion relation $k=k(\omega)=\beta\omega^3\pm\omega[\alpha(u_0^2+\alpha\omega^2/4)]^{1/2}$. We see that the dispersion curves of the excitation display two branches $k_+(\omega)$ and $k_-(\omega)$; both of them are acoustic, i.e., $k_\pm(0)=0$ (see Fig. 1).

We are interested in a possible SHG for the excitations created on the cw background. A necessary condition for the

*Corresponding author.

SHG is to satisfy the phase-matching condition $\omega_2=2\omega_1$, $k_2=2k_1$, where ω_1 and k_1 (ω_2 and k_2) are the frequency and wave vector of the fundamental (second-harmonic) wave. We find that this is indeed possible if we choose the point (ω_1,k_1) from the curve $k_+(\omega)$ and the point (ω_2,k_2) from the curve $k_-(\omega)$ with $2k_+(\omega_1)=k_-(2\omega_1)$, which means that the fundamental wave frequency should be selected as $\omega_1=(1/6\beta)[5\alpha^2+4(\alpha^4+36\alpha\beta^2u_0^2)^{1/2}]^{1/2}$.

Figure 1 shows the linear dispersion relation and the phase-matching condition for the SHG. The parameters are provided from standard single-mode optical fibers, i.e., the ZD point wavelength $\lambda_{ZD}=1.27$ μm. Near λ_{ZD}, $k^{(1)}=5\times10^{-9}$ sm^{-1}, $k^{(2)}=9\times[1.27-\lambda_0\ (\mu\text{m})]\times10^{-26}$ s^2 m^{-1}, $k^{(3)}=2.3\times\sqrt{\lambda_0\ (\mu\text{m})}[\lambda_0\ (\mu\text{m})-1]\times10^{-40}$ s^3 m^{-1}, the Kerr coefficient $n_2=1.2\times10^{-22}$ (m/V^2). The wavelength of the carrier wave and the pulse duration of the electric field are chosen as $\lambda_0=1.064$ μm and $T=10^{-12}$ s, respectively. Thus we get $\alpha=1.8\times10^{-6}$, $\beta=5.06\times10^{-10}$, $\omega_1=2.1345\times10^4$ s^{-1}, and $k_1=10.5$ m^{-1}. In the figure the dimensionless amplitude of the electric-field background is taken as $u_0=1.25$, which corresponds to the dimensional electric field $E_0=2(|k^{(1)}|c/T\omega_0 n_2)^{1/2}u_0=1.2\times10^4$ V/m when taking the effective cross area A_{eff} of the fiber as 20 μm^2, where c is the speed of light in vacuum and $\omega_0=2\pi c/\lambda_0$.

We now derive the nonlinear envelope equations controlling the SHG. By introducing the asymptotic expansion $a=\varepsilon a^{(1)}+\varepsilon^2a^{(2)}+\varepsilon^3a^{(3)}+\cdots$, $\phi=\varepsilon\phi^{(1)}+\varepsilon^2\phi^{(2)}+\varepsilon^3\phi^{(3)}+\cdots$, with ε being a small ordering parameter and a and ϕ being the functions of the fast variables (z,t) and the slow variables $(\varepsilon z,\varepsilon t)$, Eqs. (2) and (3) are transformed into

$$a_z^{(j)}-\beta a_{ttt}^{(j)}-(\alpha/2)\phi_{tt}^{(j)}=m^{(j)}, \tag{4}$$

$$2u_0^2a^{(j)}-(\alpha/2)a_{tt}^{(j)}-\phi_z^{(j)}+\beta\phi_{ttt}^{(j)}=n^{(j)}. \tag{5}$$

The explicit expressions of $m^{(j)}$ and $n^{(j)}$ $(j=1,2,\ldots)$ are omitted here.

In the leading order $(j=1)$, Eqs. (4) and (5) are linear equations which admit the solution $\phi^{(1)}=\phi_{1j}\exp(i\theta_j)+\text{c.c.}$ and $a^{(1)}=a_{1j}\exp(i\theta_j)+\text{c.c.}$ with $\theta_j=\omega_jt-k_jx$. In the case of SHG we take the leading solution as a superposition of two components (i.e., the fundamental and the second harmonic waves):

$$\phi^{(1)}=\phi_{11}\exp(i\theta_1)+\phi_{12}\exp(i\theta_2)+\text{c.c.}, \tag{6}$$

$$a^{(1)}=a_{11}\exp(i\theta_1)+a_{12}\exp(i\theta_2)+\text{c.c.}, \tag{7}$$

where c.c. represents the corresponding complex conjugate, and the frequency and wave vector of the fundamental [second-harmonic] wave (k_1,ω_1) $[(k_2,\omega_2)]$ are selected according to the phase-matching condition shown in Fig. 1. The envelopes $a_{1j}=ib_j\phi_{1j}$ with $b_j=-\alpha\omega_j/(k_j-\beta\omega_j^3)$ $(j=1,2)$ are functions of the slow variables εz and εt.

In the next order, we can get the wave equations governing the envelopes of the fundamental and second-harmonic waves. For simplicity we first consider a simple case called the quasistationary approximation, i.e., the fundamental and the second-harmonic waves are infinite plane waves with their envelopes being slowly modulated with the coordinate z. Making the transformation $\varepsilon\phi_{11}=\phi_1,\varepsilon\phi_{12}=\phi_2$, the envelope equations read

$$\frac{\partial\phi_1}{\partial z}=\lambda_1\phi_1^*\phi_2\exp(-i\Delta kz), \tag{8}$$

$$\frac{\partial\phi_2}{\partial z}=\lambda_2\phi_1^2\exp(i\Delta kz), \tag{9}$$

where $\Delta k=2k_1-k_2$ is the phase mismatch and the coefficients λ_1,λ_2 are given by

$$\lambda_1=\frac{1}{2(k_1-\beta\omega_1^3)}\Bigg[-\left(2u_0^2+\frac{\alpha\omega_1^2}{2}\right)\left(-\alpha\omega_1\omega_2b_1+\frac{\alpha\omega_2^2}{2}b_1+\alpha\omega_1\omega_2b_2+\frac{\alpha\omega_1^2}{2}b_2+3\beta\omega_1^2\omega_2-3\beta\omega_1\omega_2^2\right)$$
$$+(-k_1+\beta\omega_1^3)[(k_2+3\beta\omega_1^2\omega_2+3\beta\omega_1\omega_2^2-\beta\omega_2^3)b_1-\alpha\omega_1\omega_2-6u_0^2b_1b_2-(k_1-3\beta\omega_1\omega_2^2-3\beta\omega_1^2\omega_2+\beta\omega_1^3)b_2]\Bigg],$$
$$\lambda_2=\frac{1}{2(k_2-\beta\omega_2)}\left[\left(2u_0^2+\frac{\alpha\omega_2^2}{2}\right)\left(\frac{3}{2}\alpha b_1\omega_1^2-3\beta\omega_1^3\right)+(-k_2+\beta\omega_2^3)\left(k_1b_1-5\beta\omega_1^3b_1+\frac{\alpha\omega_1^2}{2}\right)\right]. \tag{10}$$

The envelope equations (8) and (9) can be solved exactly [7]. Let $\phi_1=f\exp(i\varphi_f)$ and $\phi_2=h\exp(i\varphi_h)$ with f and φ_h two real functions; Eqs. (8) and (9) become $\partial f/\partial z=\lambda_1fh\cos\theta$, $\partial h/\partial z=\lambda_2h^2\cos\theta$, $f\partial\varphi_f/\partial z=\lambda_1gh\sin\theta$, and $h\partial\varphi_h/\partial z=-\lambda_2f^2\sin\theta$ with the relative phase angle defined by $\theta=\varphi_h-2\varphi_f+\Delta kz$. One of the conservative quantities for these equations reads $f^2/\lambda_1-h^2/\lambda_2=m$, where m is the integration constant. We can see λ_1 and λ_2 determine the rate of energy transfer between the fundamental wave and the second-harmonic wave. Another conservative quantity is given by $\Gamma_h=-\Delta kh^2/2+\lambda_2f^2h\sin\theta$. With these relations we obtain

$$\int_{z_1}^{z}dz=\frac{1}{2}\int_{H(z_1)}^{H(z)}\frac{dh^2}{\{\lambda_2^2[m+(\lambda_1/\lambda_2)h^2]^2h^2-(\Gamma_h+\Delta kh^2)^2/2\}^{1/2}} \tag{11}$$

where $H(z)\equiv h^2(z)$. The integral equation (11) gives the general solution $H(z)$ at the distance z for arbitrary inputs $F(z_1)$ $[F(z)\equiv f^2(z)]$, $H(z_1)$ at z_1. If the power of the initial second-harmonic wave is zero, i.e., $H(z_1)=0$ and hence leading to $\Gamma_h=0$, the integral (11) is simplified as $\int_{z_1}^{z}dz=(1/2)\int_0^{H(z)}dh^2\{\lambda_2^2[F(z_1)+\lambda_1h^2/\lambda_2]^2h^2-(\Delta kh^2/2)^2\}^{-1/2}$,

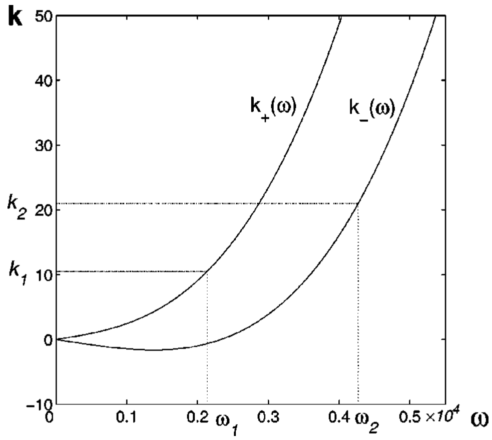

FIG. 1. The linear dispersion relation and phase matching of a SHG for the excitation on a cw background. The phase-matching condition can be satisfied if (ω_1,k_1) and (ω_2,k_2) are chosen from different dispersion branches $k_+(\omega)$ and $k_-(\omega)$, respectively. The parameters used in the figure are $\alpha=1.854\times10^{-6}$, $\beta=5.0612\times10^{-10}$, $u_0=1.25$.

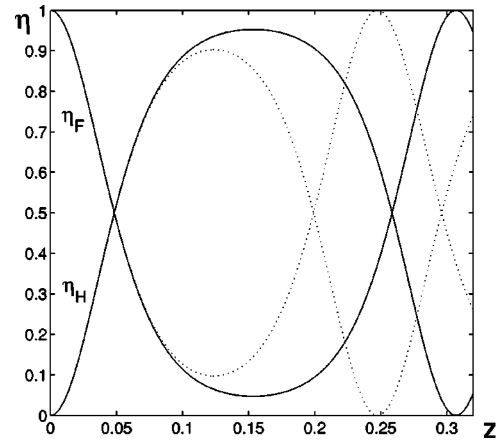

FIG. 2. The energy conversion efficiency between the fundamental and the second-harmonic waves with the system parameters $\alpha=1.854\times10^{-6}$, $\beta=5.061\times10^{-10}$, $u_0=1.25$, $H(0)=0$, and phase mismatch $\Delta k=1.0$ (bold curve). The dashed curves show the effect of increasing phase mismatch $\Delta k=3.0$.

where $F(z_1)=m$ is the initial power of the fundamental wave. When λ_1 and λ_2 have different signs, the general expression for the magnitude of the second-harmonic wave reads

$$H(z)=-(\lambda_2/\lambda_1)F(z_1)B_s^2\,\mathrm{sn}^2([-\lambda_1\lambda_2F(z_1)A_s^2z^2],\gamma_s), \quad (12)$$

where γ_s is the modulus of the elliptic function sn, given by $\gamma_s=A_{s-}^2/A_{s+}^2$ with $A_{s\pm}^2=\{(2-\sigma)\pm[(2-\sigma)^2-4]^{1/2}\}/2$, where $\sigma=\Delta k/2)^2/[F(z_1)\lambda_1\lambda_2]$ is responsible for the properties of the fiber. The results for the energy conversion efficiency of the second-harmonic wave, $\eta_H=H(z)/F(z_1)$, and the fundamental wave, $\eta_F=F(z)/F(z_1)$, have been plotted in Fig. 2, from which we see that there is a periodic energy conversion between two wave modes. The bold curves in the figure show the generation of η_H with phase mismatch $\Delta k=1.0$ from initial value of $H(0)=0$. The dashed curves show the effect of the increment of $\Delta k=3.0$. It is clear that the energy conversion efficiency decreases with increasing phase mismatch Δk. In the ideal case with zero phase matching, i.e., $\Delta k=0$, Eq. (12) becomes $H=-(\lambda_2/\lambda_1)F(z_1)\tanh^2\times\{[-\lambda_1\lambda_2\,F(z_1)z^2]^{1/2}\}$. In this situation there is no back conversion and the maximal conversion efficiency η_H can approach 97.9%.

The theoretical prediction of the SHG in optical fibers presented above is verified numerically. The analytical solution obtained above is taken as an initial condition. We use a split-step Fourier transform method to integrate Eq. (1) directly. We assume that when the optical field propagates over a small distance, the dispersive and nonlinear effects act independently. In the first step, the nonlinearity acts alone; thus one has $u'(z,t)=u(0,t)\exp[i|u(0,t)|^2z]$. In the second step the dispersion acts alone, so we have $u(z,t)=F^{-1}(\exp\{[(i/2)\alpha\omega^2-i\beta\omega^3]z\}F[u'(z,t)])$, where F^{-1} denotes an inverse Fourier transformation and ω is the frequency in the Fourier domain. We use a fast Fourier transform algorithm to obtain numerical evaluation of $u(z,t)$. Because practically the actual background shape of a cylindrical optical beam is a super-Gaussian, we assume that the background has the form $u_0\exp\{-[(z^2+t^2)/L^2]^8\}$. In our simulation, we use $u_0=1.25$, $L=75.0$, and $f(0)=4.0\times10^{-2}$.

Shown in Fig. 3 is our numerical result. We see that initially there is only a fundamental wave. Near $z=10$ a second-harmonic wave appears and its amplitude becomes larger and larger as distance (z) increases. There is obvious energy conversion between the fundamental wave and the second-harmonic wave. This behavior agrees well with the analytical result. In addition the numerical solution shows a nice stability and hence is promising for a practical experimental observation.

We also get the frequency spectrum using the Fourier transform of $u(z,t)$. Shown in Fig. 4 is the evolution of the frequency spectrum in the SHG process. For clearness the contribution of the super-Gaussian background has been suppressed. The fundamental wave (with frequency $\omega=2134.5$) and the second-harmonic wave (with frequency $\omega=4269$) can be seen clearly. Due to the higher-order nonlinearities of

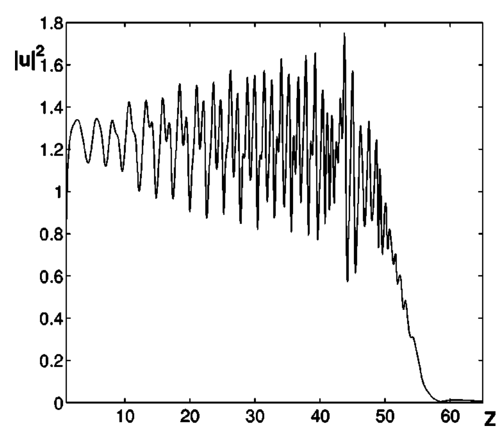

FIG. 3. The second-harmonic wave generated on a super-Gaussian background in nonlinear optical fibers. The parameters α,β,u_0 are the same as those in Fig. 2 with $L=75.0$, $f(0)=4\times10^{-2}$.

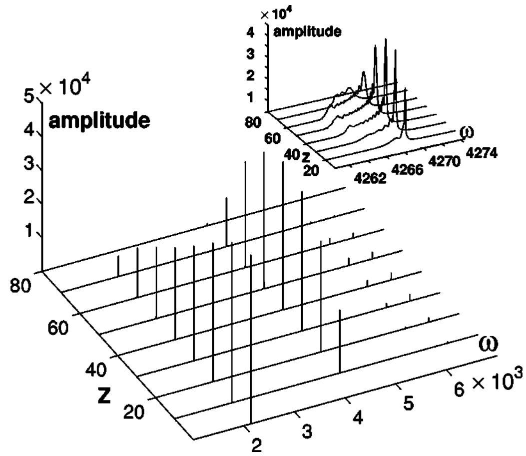

FIG. 4. Evolution of the frequency spectrum in the SHG process on a super-Gaussian background. The fundamental wave (with frequency $\omega=2134.5$) and the second-harmonic wave (with frequency $\omega=4269$) can be seen clearly. There appear small-amplitude third- and higher-harmonic waves due to the higher-order nonlinearities of the system. The parameters α, β, u_0 are chosen the same as in Fig. 2. The inset is the frequency spectrum near $\omega=4269$.

the system, small-amplitude third- and higher-harmonic waves appear. The parameters α, β, u_0 are chosen the same as in Fig. 2. The inset shows the detailed structure of the frequency spectrum near $\omega=4269$.

Note that the quasistationary approximation used in deriving Eqs. (8) and (9) is valid only for infinitely large plane waves. For narrower excitations the group velocity mismatch between the fundamental and the second-harmonic waves should be taken into account. Using a similar approach the same as that for deriving the Eqs. (8) and (9) but now assuming that the envelopes depend also on the slowly varying time variable εt, we obtain

$$(\partial \phi_1/\partial z) + (1/v_{g1})(\partial \phi_1/\partial t) = \lambda_1 \phi_1^* \phi_2 \exp(-i\Delta kz), \tag{13}$$

$$(\partial \phi_2/\partial z) + (1(v_{g2})(\partial \phi_2/\partial t) = \lambda_2 \phi_1^2 \exp(i\Delta kz), \tag{14}$$

where $v_{g1}=d\omega_1/dk_1$ and $v_{g2}=d\omega_2/dk_2$ are the group velocities of the fundamental wave and the second-harmonic wave, respectively. λ_j ($j=1,2$) are the same as those given by Eq. (10). If we use z and $\eta=t-z/v_{1g}$ instead of z and t as independent variables, the coupled amplitude equations Eqs. (13) and (14) under the phase-matching condition $\Delta k=0$ become

$$(\partial \phi_1/\partial z) = \lambda_1 \phi_1^* \phi_2, \tag{15}$$

$$(\partial \phi_2/\partial z) + \nu(\partial \phi_2/\partial \eta) = \lambda_2 \phi_1^2, \tag{16}$$

with $\nu=1/v_{g2}-1/v_{g1}$. The walk-off parameter ν indicates the separation between the two pulses. If at $z=0$, $\phi_1(t)=A/(1+t^2/\tau^2)$, the solution of Eqs. (15) and (16) has the form [7]

$$\phi_1 = \sqrt{\frac{1}{\lambda_1\lambda_2}}\frac{A}{(1+\widetilde{\eta})^{1/2}[1+(\widetilde{\eta}-\widetilde{z})^2]}\left\{\cosh\xi + \frac{\widetilde{\eta}}{f}\sinh\xi\right\}, \tag{17}$$

$$\phi_2 = -\frac{A\tau_{cr}}{\tau\lambda_1[1+(\widetilde{\eta}-\widetilde{z})^2]}\left\{\frac{\widetilde{z}\cosh\xi + [f-\widetilde{\eta}(\widetilde{\eta}-\widetilde{z}/f)]\sinh\xi}{\cosh\xi + (\widetilde{\eta}/f)\sinh\xi}\right\} \tag{18}$$

with $\widetilde{\eta}=\eta/\tau$, $\widetilde{z}=z/l_\nu$, $\tau_{cr}=\nu/A$, $f=(\tau^2/\tau_{cr}^2-1)^{1/2}$, $\xi=f[\tanh^{-1}\widetilde{\eta}-\tanh^{-1}(\widetilde{\eta}-\widetilde{z})]$, where $l_\nu=\tau/\nu$ is the propagating distance over which the overlapping fundamental and second-harmonic pulses of width τ are clearly separated.

In conclusion, we have investigated the resonant interaction between two exciting waves generated on a cw background in nonlinear optical fibers. We have shown that in normal dispersion regime and near the zero-dispersion point of a single-mode optical fiber the phase-matching condition of a SHG can be satisfied by a suitable selection of the wave vectors and frequencies of the fundamental and the second-harmonic waves. Thus a different type of second-harmonic generation with a different physical mechanism is possible in optical fibers. Using a multiscale method we have derived the nonlinearly coupled envelope equations for the SHG and their explicit solutions are presented and checked numerically. Note that the SHG in optical fibers suggested in the present work is based on a different mechanism and hence does not need any breaking of the centrosymmetry of the system. Because experimentally an infinitely extended cw background is not realizable, to observe the SHG predicted here one should consider the excitations on a background with a large but finite extent, as in the case of the observation for dark solitons in optical fibers [8].

This work was supported by the Chinese National Key Basic Research Special Fund (CNKBRSF), by the National Natural Science Foundation of China (NSFC), by the Hong Kong Research Grants Council (RGC), and by a Hong Kong Baptist University Faculty Research Grant (FRG).

[1] Y. Fujii *et al.*, Opt. Lett. **5**, 48 (1980); Y. Ohmori and Y. Sasaki, Appl. Phys. Lett. **39**, 466 (1981).
[2] B. P. Antonyuk *et al.*, Opt. Commun. **147**, 143 (1998).
[3] U. Osterberg and W. Margulis, Opt. Lett. **11**, 516 (1986).
[4] G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. (Academic, New York, 1995).
[5] Y. S. Kivshar and B. Luther-Davies, Phys. Rep. **298**, 81 (1998), Sec. 3.4.
[6] Guoxiang Huang and M. G. Velarde, Phys. Rev. E **54**, 3048 (1996).
[7] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984).
[8] A. Hasgawa and Y. Kodama, *Solitons in Optical Communications* (Clarendon, Oxford, 1995), Chap. 16.